

EFFECTS OF INJECTION AND SUCTION ON PRESSURE
DISTRIBUTION IN POROUS WALL CHANNELS

A THESIS

Presented to
The Faculty of the Graduate Division

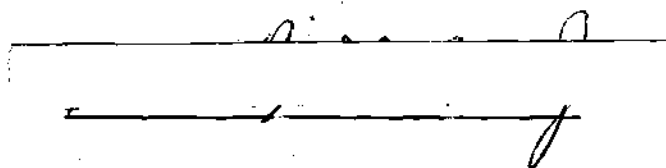
by
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Of the Requirements for the Degree
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DISTRIBUTION IN POROUS WALL CHANNELS

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SUMMARY

The effect of injection and suction through a porous-wall channel* having a rectangular cross-section is investigated in detail by the solution of the Navier-Stokes equations. The pressure distribution is obtained for two-dimensional, fully developed, steady-state, laminar flow of an incompressible fluid with uniform suction or injection of the same fluid at the walls.

The solution to the Navier-Stokes equation leads to a third-order nonlinear differential equation in (λ) with the appropriate boundary conditions. For small suction or injection Reynolds numbers, this latter equation is solved by a perturbation method. First-order expressions in (λ) for the dependence of the pressure distribution and velocity on position coordinates, channel dimensions, and fluid properties are obtained.

The pressure drop in the direction of the main stream is found to be appreciably less for suction and appreciably greater for injection in the porous-wall channel than that for a solid-wall channel having the same dimensions and the same entrance Reynolds number.

It is noted that a small value of suction ($R = 1.0$, $\alpha = 1.0$, and $N_{Re} = 500$) may decrease the pressure drop by 86.4%. A small value of injection ($R = -1.0$, $\alpha = 1.0$, and $N_{Re} = 500$) may increase the pressure drop by 148.1%.

* A porous-wall channel is defined as one in which the suction and/or injection velocity is much less than the velocity of the main stream.

NOMENCLATURE

| | |
|--------------------|--|
| $f(\lambda)$ | function defined by Equation (22) |
| $2h$ | channel width |
| k | integration constant |
| N_{Re} | $= 4\bar{u}(0)h/\nu$, main stream Reynolds number |
| $P(\eta, \lambda)$ | pressure at a point in the channel |
| R | $= Vh/\nu$, injection or suction Reynolds number |
| $\bar{u}(\eta)$ | average velocity in x-direction |
| $\bar{u}(0)$ | average velocity of main stream at $x = 0$ |
| $u(\eta, \lambda)$ | velocity of main stream in the x-direction |
| V | velocity at top plate in y-direction |
| αV | velocity at bottom plate in y-direction |
| $v(\eta, \lambda)$ | velocity of main stream in the y-direction |
| x | distance measured parallel to wall |
| y | distance measured from the centerline of the channel to the walls |
| α | a constant |
| λ | dimensionless variable defined by Equation (4) |
| η | dimensionless variable defined by Equation (4) |
| ρ | density |
| μ | dynamic viscosity |
| ν | kinematic viscosity |
| τ | shear stress in a porous wall channel |

NOMENCLATURE (Continued)

τ_o shear stress in a solid wall channel
 ψ stream function

CHAPTER I

INTRODUCTION

Scope of Work

The laminar incompressible flow of a fluid in a two-dimensional channel with two porous walls is a problem involving diffusion phenomena. In order to solve for the pressure distribution, one must know the magnitude and direction of the velocity at any point in the channel.

Schlichting^{1*} has devoted considerable theoretical effort to the boundary-layer flow over a porous flat plate. However, Schlichting used the boundary-layer equations which are approximations to the Navier-Stokes equations and are not applicable to fully developed channel flow. Olson² and Van Der Hegge Zijnen³ investigated the problem by relating the over-all pressure drop to the flow rate in a channel with porous walls; however, the problem was simplified by restricting the discussion to a one-dimensional flow and neglecting kinetic energy effects. Research has been done by A. S. Berman⁴ which relates the overall pressure drop to the flow rate with equally porous walls. However, the scope of this work was limited to equal suction velocities at the walls of the channel; injection was not considered.

The flow through porous channels is of interest in certain fluid flow problems. For instance, when a fluid flows down a channel, problems which arise from a large pressure drop may be overcome by suction of fluid through part of the wall. Certain types of combustion-chambers

*Superscripts refer to references contained in the Bibliography.

may be cooled by injection of fluid at the walls. Special dehydration processes may be examined by considering injection of fluid into a main stream flowing between an infinite porous flat plate and an infinite solid flat plate.

For these reasons, it seemed desirable to investigate the possibility of obtaining a general solution to the Navier-Stokes equations for channel flow when the fluid is injected or withdrawn at different velocities at the top and bottom plates.

CHAPTER II

DEVELOPMENT

Reduction of the Flow Equations

Consider the flow of a fluid between two infinite parallel plates; the distance between the plates is taken to be $2h$.

A coordinate system is chosen with the origin at the center of the channel. The y -axis is perpendicular to the walls and the x -axis is in a plane parallel to the walls as shown in Figure 1. All properties in the z -direction (perpendicular to the x - y plane) are constant.

Assume the top and bottom plates have different but uniform permeabilities. Let the suction velocity at the top plate be αV , $-1 < \alpha \leq 1$,* and the suction velocity at the bottom plate by V .

The following flow conditions are imposed:

- (1) A steady two-dimensional flow prevails.
- (2) Incompressible flow.
- (3) No external forces act on the fluid.
- (4) The flow is laminar.
- (5) The velocity of the fluid leaving the walls of the channel is independent of x .
- (6) For injection, the fluid being injected is the same as that in the main stream.
- (7) The velocity profile of the main stream is fully developed at $x = 0$ where suction or injection is initiated.

*See Chapter V.

(8) The velocity profiles are similar.

Under the assumed conditions and choice of axes (see Figure 1), the Navier-Stokes equations as developed by Schlichting¹ reduce to

$$\frac{u}{h} \frac{\partial u}{\partial \eta} + \frac{v}{h} \frac{\partial u}{\partial \lambda} = - \frac{1}{\rho h} \frac{\partial P}{\partial \eta} + \frac{v}{h^2} \left(\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \lambda^2} \right) \quad (1)$$

$$\frac{u}{h} \frac{\partial v}{\partial \eta} + \frac{v}{h} \frac{\partial v}{\partial \lambda} = - \frac{1}{\rho h} \frac{\partial P}{\partial \lambda} + \frac{v}{h^2} \left(\frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \lambda^2} \right) \quad (2)$$

and the continuity equation is

$$\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \lambda} = 0 \quad (3)$$

where the dimensionless variables

$$\lambda = \frac{Y}{h}, \quad \eta = \frac{X}{h} \quad (4)$$

have been introduced, and

$$u = u(\eta, \lambda)$$

$$v = v(\eta, \lambda)$$

Since the suction velocities are different at the top and bottom plates (see Figure 1), the boundary conditions are as follows:

$$u(\eta, 1) = 0 \quad (5)$$

$$u(\eta, -1) = 0 \quad (6)$$

$$v(\eta, 1) = \alpha V \quad (7)$$

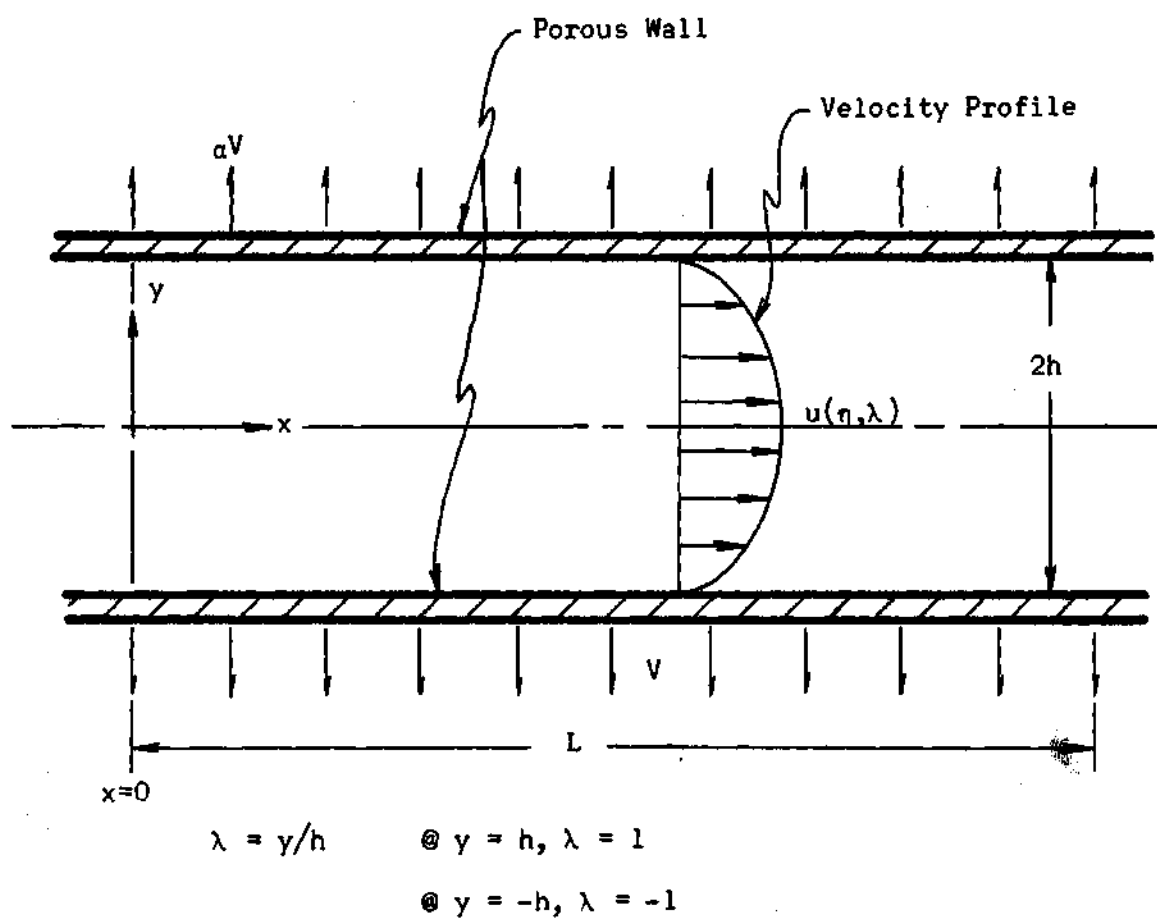


Figure 1. A Porous Wall Channel.

$$v(\eta, -1) = -V. \quad (8)$$

Selection of a Stream Function*

For two-dimensional incompressible flow, a stream function ψ exists such that

$$d\psi = \frac{\partial \psi}{\partial \eta} d\eta + \frac{\partial \psi}{\partial \lambda} d\lambda \quad (9)$$

where

$$u = \frac{\partial \psi}{\partial \lambda} \quad (10)$$

$$v = -\frac{\partial \psi}{\partial \eta}. \quad (11)$$

The stream function becomes

$$d\psi = -vd\eta + ud\lambda \quad (12)$$

Under the assumption of similar velocity profiles, one may say that

$$\frac{u(\eta, \lambda)}{\bar{u}(\eta)} = f_1(\lambda), \quad \frac{v(\eta, \lambda)}{V} = f_2(\lambda)$$

or

$$u = \bar{u}(\eta) \cdot f_1(\lambda) \quad (13)$$

$$v = Vf_2(\lambda) \quad (14)$$

Substituting equations (13) and (14) into the continuity equation (3), one obtains

* The technique for selection of a stream function was suggested by Dr. Neil R. Johnson.

$$\bar{u}'(\eta)f_1(\lambda) + Vf_2'(\lambda) = 0. \quad (15)$$

Equation (15) may be solved directly by the Separation of Variables technique. One obtains the following:

$$\frac{f_2'(\lambda)}{f_1(\lambda)} = C, \quad -\frac{\bar{u}'(\eta)}{V} = C$$

and

$$f_1(\lambda) = \frac{f_2'(\lambda)}{C} \quad (16)$$

$$\bar{u}'(\eta) = -VC \quad (17)$$

where C is the separation constant which must be real and $\neq 0$. Integrating equation (17), one obtains

$$\bar{u}(\eta) = -VC\eta + \bar{u}(0) \quad (18)$$

where $\bar{u}(0)$ is the integration constant.

Substituting equations (16) and (18) into equation (13), one obtains

$$u(\eta, \lambda) = [-CV\eta + \bar{u}(0)] \cdot \frac{f_2'(\lambda)}{C} \quad (19)$$

Substituting equations (19) and (14) into equation (12), one obtains

$$d\psi = -Vf_2(\lambda)d\eta + [-CV\eta + \bar{u}(0)] \cdot \frac{f_2'(\lambda)}{C} d\lambda$$

or

$$d\psi = \bar{u}(0) \frac{f_2'(\lambda)}{C} d\lambda - [Vf_2(\lambda)d\eta + v_\eta f_2'(\lambda)d\lambda]$$

or

$$d\psi = \frac{\bar{u}(0)}{C} f_2'(\lambda)d\lambda - d[V_\eta f_2(\lambda)] \quad (20)$$

Integrating equation (20), one obtains

$$\psi(\eta, \lambda) = \left[\frac{\bar{u}(0)}{C} - v_\eta \right] f(\lambda) \quad (21)$$

A perturbation solution^{*} was carried out using the stream function equation (21) with $C = 1$. One obtains for the average velocity^{**}

$$\bar{u}(\eta) = \frac{1}{2} (\alpha + 1) [\bar{u}(0) - v_\eta]$$

Since the average velocity is defined to be $u(0)$ at $\eta = 0$, one obtains for the separation constant $C = \frac{1}{2} (\alpha + 1)$. From equation (21) the stream function becomes

$$\psi(\eta, \lambda) = \left[\frac{2\bar{u}(0)h}{\alpha + 1} - v_\eta \right] f(\lambda) \quad (22)$$

With this suitable choice of the stream function and the given boundary conditions, the velocity components may be obtained from equations (10) and (11) as

$$u(\eta, \lambda) = \left[\frac{2\bar{u}(0)}{\alpha + 1} - v_\eta \right] f(\lambda) \quad (23)$$

$$v(\eta, \lambda) = Vf(\lambda) = v(\lambda) \quad (24)$$

* See Chapter III.

** See Chapter IV, p. 24.

In these equations $f(\lambda)$ is some function, yet to be determined, of the parameter λ . With the suitable choice of the stream function, the y-component of velocity becomes a function of λ only. Experimental results* indicate that the transverse component of velocity is not a function of the distance parameter x (or η) under the assumed flow conditions.

Cubic Velocity Assumption

At this point a simplified solution may be obtained by assuming that $f(\lambda)$ is a cubic function of the form

$$f(\lambda) = A + B\lambda + C\lambda^2 + D\lambda^3. \quad (25)$$

The boundary conditions on the function $f(\lambda)$ and its derivatives are readily obtained from equations (5) through (8) and equations (23) and (24). Thus, one obtains

$$f'(1) = 0 \quad (26)$$

$$f'(-1) = 0 \quad (27)$$

$$f(1) = \alpha \quad (28)$$

$$f(-1) = -1. \quad (29)$$

The constants A, B, C, and D in equation (25) may be evaluated from the boundary conditions, equations (26) through (28). Thus, equation (25) becomes

* See references 5 through 11.

$$f(\lambda) = \frac{1}{2} (\alpha - 1) + \frac{3}{4} (\alpha + 1) \lambda - \frac{1}{4} (\alpha + 1) \lambda^3. \quad (30)$$

Equation (30) is the exact solution for $R = 0$ and good for small R . Also, it may be noted here that if $\alpha = 1.0$, equation (30) is the first-term of Berman's solution for equal suction velocity at both walls.

A Complete Solution

In order to obtain a more complete solution, equations (23) and (24) are substituted into the equations of motion (1) and (2), the result is

$$\frac{1}{\rho h} \frac{\partial P}{\partial \eta} = \left[\frac{2\bar{u}(0)}{\alpha + 1} - v_\eta \right] \left[\frac{v}{h^2} f'''' + \frac{v}{h} (f'^2 - f f'') \right] \quad (31)$$

and

$$\frac{1}{h\rho} \frac{\partial P}{\partial \lambda} = \frac{v}{h^2} \frac{d^2 v}{d\lambda^2} - \frac{v}{h} \frac{dv}{d\lambda}. \quad (32)$$

The right-hand side of equation (32) is seen to be a function of λ only; hence, differentiation with respect to η yields

$$\frac{\partial^2 P}{\partial \eta \partial \lambda} = 0, \quad (33)$$

so that upon differentiation (31) with respect to λ , one obtains

$$\left[\frac{2\bar{u}(0)}{\alpha + 1} - v_\eta \right] \cdot \frac{d}{d\lambda} \left(\frac{v}{h^2} f'''' + \frac{v}{h} [f'^2 - f f''] \right) = 0 \quad (34)$$

If equation (34) is to be satisfied for all $x \geq 0$, then

$$\frac{d}{d\lambda} \left(\frac{v}{h^2} f''' + \frac{v}{h} [f'^2 - ff''] \right) = 0. \quad (35)$$

Integrating equation (35), one obtains

$$f''' + R[f'^2 - ff''] = k \quad (36)$$

where the suction or injection Reynolds number is

$$R = \frac{Vh}{\nu} \quad (37)$$

and k is the constant of integration. It may be noted that V is negative for injection. Hence, the injection Reynolds number is negative.

Equation (36) is a third-order, nonlinear, nonhomogeneous differential equation. With the associated boundary conditions, equations (26) through (29), the solution of equation (36) constitutes an exact solution to the equations of motion and continuity as formulated. Four boundary conditions are needed since the integration constant k is still to be determined.

A Simple Third Order Equation

If one considers the limiting form of equation (36) as R tends to zero (the order of the equation is unchanged), a simple third-order equation,

$$f'''(\lambda) = k \quad (38)$$

results. The exact solution to this equation, subject to the boundary conditions, is equation (30). If $\alpha = 1$, the solution is the well-

known Poiseuille function¹ describing flow in a rectangular channel. In view of this, equation (36) may be studied by a perturbation treatment in which the wall Reynolds number R is used as the perturbation parameter. Such a solution will be valid for sufficiently small values of R , $|R| \leq 1$ (see Chapter V), and is outlined in the next chapter.

CHAPTER III

PERTURBATION SOLUTION

Expanding the function $f(\lambda)$ and the constant of integration, k , of equation (36),

$$f(\lambda) = g_0(\lambda) + g_1(\lambda)R + g_2(\lambda)R^2 + \dots + \quad (39)$$

$$g_i(\lambda)R^i + \dots + g_n(\lambda)R^n$$

$$k = C_0 + C_1R + C_2R^2 + \dots C_iR^i + \quad (40)$$

$$\dots + C_nR^n$$

Here the $g_i(\lambda)$'s and C_i 's are taken to be independent of R . Substitution of equations (39) and (40) into (36) and collecting powers of R yields the following equations:

$$\text{Zeroth-Order: } g_0''' = C_0 \quad (41)$$

$$\text{First-Order: } g_1''' = C_1 + g_0g_0' - (g_0')^2 \quad (42)$$

$$\text{Second Order: } g_2''' = C_2 + g_0g_1' + g_0'g_1 - \quad (43)$$

$$2g_0'g_1'$$

The boundary conditions to be satisfied by the g_i 's are obtained from equations (26) through (29),

$$(1) \quad g_i'(1) = 0$$

$$(2) \quad g_1'(-1) = 0$$

$$(3) \quad g_1(1) = (\delta_{01})$$

$$(4) \quad g_1(-1) = -1(\delta_{01})$$

where Kronecker's δ has been used.

Equations (41), (42), (43), and higher orders are ordinary linear third-order equations which are solved to give successive approximations to $f(\lambda)$. The solutions* of equations (41), (42), and (43) are as follows:

Zeroth Order:

$$g_0(\lambda) = \frac{1}{2}(a-1) + \frac{3}{4}(a+1)\lambda - \frac{1}{4}(a+1)\lambda^3$$

First-Order:

$$\begin{aligned} g_1(\lambda) = & \left[\frac{1}{16}(a+1) - \frac{1}{32}(a+1)^2 \right] - \frac{1}{560}(a+1)^2\lambda + \\ & \left[\frac{1}{16}(a+1)^2 - \frac{1}{8}(a+1) \right]\lambda^2 + \frac{3}{1120}(a+1)^2\lambda^3 + \\ & \left[\frac{1}{16}(a+1) - \frac{1}{32}(a+1)^2 \right]\lambda^4 - \frac{1}{1120}(a+1)^2\lambda^7 \end{aligned}$$

Second Order:

$$\begin{aligned} g_2(\lambda) = & S + R\lambda + \frac{1}{2}Q\lambda^2 + \frac{1}{6}C\lambda^3 + \frac{1}{24}D\lambda^4 \\ & + \frac{1}{60}J\lambda^5 + \frac{1}{120}K\lambda^6 + \frac{1}{210}L\lambda^7 + \frac{1}{336}M\lambda^8 \\ & + \frac{1}{504}N\lambda^9 + \frac{1}{990}P\lambda^{11} \end{aligned}$$

* See Appendix A.

where,

$$S = -\frac{3}{8} \gamma + \frac{3}{4} n\gamma + \frac{107}{280} \beta\delta$$

$$R = -\frac{1}{5} \beta + \frac{2}{5} \beta\delta + \frac{35}{99} \gamma\delta$$

$$Q = 2\gamma - 4n\gamma - \frac{37}{35} \beta\delta$$

$$C_2 = \frac{12}{5} \beta + \frac{42}{11} \delta\gamma - \frac{180}{35} \beta\delta$$

$$H = 4\beta + 12n\gamma - 8\beta\delta$$

$$I = -182 + 36n\gamma + 6\beta\delta$$

$$J = -12\beta + 24\beta\delta$$

$$K = 4\beta\delta$$

$$L = 18\gamma\delta$$

$$M = 6\beta\delta + 42\gamma - 84\gamma\delta$$

$$N = -84\gamma\delta$$

$$P = 6\gamma\delta$$

and

$$\delta = \frac{1}{4}(a + 1)$$

$$\gamma = \frac{1}{70} \delta^2$$

$$\beta = \frac{1}{4} \delta - \frac{1}{2} \delta^2$$

Clearly, further solutions such as the third and higher orders could be obtained, but they would be extremely complicated, especially since α is not specified. When an α is specified, the coefficients of the λ 's simplify greatly. For example, when $\alpha = 1$ the first, second, and zero solutions are:

Zeroth-Order:

$$g_0(\lambda) = \frac{3}{2}\lambda - \frac{1}{2}\lambda^3$$

First-Order:

$$g_1(\lambda) = \frac{1}{280}(3\lambda^3 - 2\lambda - \lambda^7)$$

Second-Order:

$$g_2(\lambda) = \frac{3}{280}\left(\frac{1}{990}\lambda^{11} - \frac{1}{36}\lambda^9 + \frac{1}{70}\lambda^7 + \frac{146}{2310}\lambda^3 - \frac{703}{14990}\lambda\right)$$

Combining the zeroth and first-order equations, one obtains the following for the first-order perturbation solution for small R ,

$$f(\lambda) = \frac{1}{2}(\alpha - 1) + \frac{3}{4}(\alpha + 1)\lambda - \frac{1}{4}(\alpha + 1)\lambda^3 + \quad (44)$$

$$R \left\{ \left[\frac{1}{16}(\alpha + 1) - \frac{1}{32}(\alpha + 1)^2 \right] - \frac{1}{560}(\alpha + 1)^2\lambda + \left[\frac{1}{16}(\alpha + 1)^2 - \frac{1}{8}(\alpha + 1) \right]\lambda^2 + \frac{3}{1120}(\alpha + 1)^2\lambda^3 + \left[\frac{1}{16}(\alpha + 1) - \frac{1}{32}(\alpha + 1)^2 \right]\lambda^4 - \frac{1}{1120}(\alpha + 1)^2\lambda^7 \right\}$$

$$k = -\frac{3}{2}(\alpha + 1) + \frac{324}{35}\left[\frac{1}{16}(\alpha + 1)^2\right]R . \quad (45)$$

Since the coefficient of the R term in equation (44) seemed large, the second-order perturbation term was calculated for various values of R , α , and λ . It was found that the R^2 term in equation (44) was less than 1% of the value of $f(\lambda)$ for $-1 < \alpha \leq 1$ and $-1 \leq R \leq 1$.*

Thus, equation (44) and (45) are solutions to equation (26), to a good approximation, as long as R is also small*. Equation (44) can be substituted into equations (23) and (24) to give the approximated velocity distributions for laminar flow in porous wall channels with different injection and suction velocities.

CHAPTER IV

THE EFFECT OF POROUS WALLS ON THE PRESSURE DISTRIBUTION

Velocity Components

Equations (44) and (45) are the first-order perturbation solutions for $f(\lambda)$ and k .

When equation (24) and (44) are combined and rearranged, the y-component of the velocity becomes

$$v(\lambda) = v \left\{ \frac{1}{2}(\alpha - 1) + (\alpha + 1)(3 - \lambda^2)\left(\frac{\lambda}{4}\right) + \right. \quad (46)$$

$$(\alpha + 1) \frac{R}{16} \left[1 - \frac{1}{2}(\alpha + 1) \right] - (\alpha + 1)\left(\frac{R}{560}\right)\lambda[(\alpha + 1) -$$

$$70\left[\frac{1}{2}(\alpha + 1) - 1\right]\lambda - \frac{3}{2}(\alpha + 1)\lambda^2 -$$

$$\left. 35\left[1 - \frac{1}{2}(\alpha + 1)\right]\lambda^3 + \frac{1}{2}(\alpha + 1)\lambda^6 \right\}.$$

The first derivative of equation (44) along with equation (23) yields the following expression for the x-component of velocity

$$u(\eta, \lambda) = \left[\frac{\bar{u}(0)}{\frac{1}{2}(\alpha + 1)} - v(\eta) \right] \left(\frac{3}{4} \right) (\alpha + 1) (1 - \lambda^2) \left\{ 1 - \right. \quad (47)$$

$$\frac{R}{840} \left[2(\alpha + 1) - 280\left[\frac{1}{2}(\alpha + 1) - 1\right]\lambda - \right.$$

$$\left. \left. 7(\alpha + 1)\lambda^2 - 7(\alpha + 1)\lambda^4 \right] \right\}.$$

From equation (47) it is seen that the velocity profile is flatter than the Poiseuille parabola and steeper close to the channel

walls. The maximum velocity in the direction of the main stream does not necessarily occur at the channel center $\lambda = 0$.

The Pressure Distribution

The pressure distribution in the channel is readily obtained from equation (31). Thus, multiplying equation (31) by $(+h^2/v)$ one obtains

$$\frac{h}{\rho v} \frac{\partial P}{\partial \eta} = \left[\frac{\bar{u}(0)}{\frac{1}{2}(\alpha + 1)} - v\eta \right] \left\{ R[f'^2 - ff''] + f''' \right\} \quad (48)$$

Using equation (36) and rearranging gives

$$\frac{\partial P}{\partial \eta} = \frac{\mu}{h} \left[\frac{\bar{u}(0)}{\frac{1}{2}(\rho + 1)} - v\eta \right] \quad (49)$$

Further from equation (32),

$$\frac{\partial P}{\partial \lambda} = \frac{\mu}{h} \frac{d^2 v}{d\lambda^2} - \rho v \frac{dv}{d\lambda} \quad (50)$$

Integrating equations (49) and (50) and making use of equation (46), one obtains

$$p(\eta, \lambda) = p(0, 0) - \rho v^2 [f^2 - f^2(0)] + \quad (51)$$

$$\frac{\mu v}{h} [f'(\lambda) - f'(0)] + \frac{\mu}{h} \left[\frac{\bar{u}(0)}{\frac{1}{2}(\alpha + 1)} \eta - v\eta^2 \right]$$

Using the first-order results for $f(\lambda)$ and its derivative one can evaluate the pressure distribution.

From equation (51) the pressure drop in the major flow direction

is

$$p(0,\lambda) - p(\eta,\lambda) = \frac{-k\mu}{h} \left[\frac{\bar{u}(0)}{\frac{1}{2}(\alpha + 1)} \eta - v\eta^2 \right] . \quad (52)$$

In dimensionless form, equation (52) becomes

$$\frac{p(0,\lambda) - p(\eta,\lambda)}{\frac{1}{2} \rho \bar{u}(0)^2} = \frac{12(\alpha + 1)}{N_{Re}} \left[2 - \frac{27(\alpha + 1)R}{35} \right] \left[\frac{1}{\alpha + 1} - \frac{R}{N_{Re}} \cdot \eta \right] \eta \quad (53)$$

where the entrance Reynolds number has been introduced as

$$N_{Re} = \frac{4h \bar{u}(0)}{v} \quad (54)$$

and the first-order value for k has been used.

Average Velocity

The average velocity of the main stream at any point x along the channel is defined as

$$\bar{u}(\eta) \equiv \frac{\int_{-1}^1 u(\eta,\lambda) d\lambda}{2} . \quad (55)$$

Combining equation (47) and (55) and performing the indicated operations, one obtains

$$\bar{u}(x) = \frac{1}{2}(\alpha + 1) \left[\frac{\bar{u}(0)}{\frac{1}{2}(\alpha + 1)} - v\eta \right] \quad (56)$$

At this point, it may be noted that by dividing equation (47) by equation (56) one obtains

$$\frac{u(\eta, \lambda)}{\bar{u}(\eta)} = \frac{f'(\lambda)}{\frac{1}{2}(\alpha + 1)} \quad (57)$$

which is independent of the quantity η .

The Shear Stress

The shear stress at the wall of the channel is defined as

$$\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_w = \mu \frac{\partial u}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial y} \bigg|_w \quad (58)$$

From equation (4),

$$\frac{\partial \lambda}{\partial y} = \frac{1}{h} \quad (59)$$

From equation (23),

$$\frac{\partial u}{\partial \lambda} = \left[\frac{2\bar{u}(0)}{\alpha + 1} - v_\eta \right] f''(\lambda) \quad (60)$$

The function $f''(\lambda)$ may be obtained from equation (44), the first order perturbation solution.

$$f''(\lambda) = -\frac{3}{4}(\alpha + 1) \left[2\lambda - \frac{R}{840} \left\{ 280 \left[\frac{1}{2}(\alpha + 1) - 1 \right] + 18(\alpha + 1)\lambda - 840 \left[\frac{1}{2}(\alpha + 1) - 1 \right] - 42(\alpha + 1)\lambda^5 \right\} \right] \quad (61)$$

Combining equations (58), (59), (60), and (61), one obtains for the shear stress

$$\tau = \frac{\mu}{h} \left[\frac{2\bar{u}(0)}{a+1} - v\eta \right] \left(-\frac{3}{4} \right) (a+1) \left[2\lambda - \right. \quad (62)$$

$$\left. \frac{R}{840} \left\{ 280 \left[\frac{1}{2}(a+1) - 1 \right] + 18(a+1)\lambda - \right. \right. \\ \left. \left. 840 \left[\frac{1}{2}(a+1) - 1 \right] \lambda^2 - 42(a+1)\lambda^5 \right\} \right]$$

When $R = 0.0$, the shear stress in a solid wall channel may be obtained from equation (62) as

$$\tau_o = - \frac{\mu}{h} \left[\frac{2\bar{u}(0)}{a+1} \right] \frac{3}{2} (a+1) \lambda \quad (63)$$

Dividing equation (62) by equation (63) and evaluating the resulting ratio at the wall of the channel, $\lambda = 1.0$, one obtains

$$\frac{\tau}{\tau_o}_{\lambda=1.0} = \frac{(a+1)}{2} \left[\frac{1}{a+1} - \frac{2R}{N_{Re}} \cdot \eta \right] \left[2 - \frac{R}{840} (256(a+1) - 560) \right] \quad (64)$$

Equation (64) is the ratio of the shear stress in a porous wall channel to that in a solid wall channel evaluated at the wall $\lambda = 1.0$.

CHAPTER V

RANGES OF VALIDITY

The range of α is restricted to $-1 < \alpha \leq 1$. All possible cases may still be handled since the larger suction or injection velocity may be assigned to the bottom plate.

It may be noted that in all the previous equation R , the injection or suction Reynolds number, carries a sign to distinguish between injection and suction. When $R < 0$ injection occurs and when $R > 0$ suction occurs.

By examining equation (56) for suction,

$$\bar{u}(\eta) = \frac{1}{2}(\alpha + 1) \left[\frac{2\bar{u}(0)}{\alpha + 1} - V \cdot \eta \right] \quad (56)$$

the values of (η) are limited by the fact that at some value of x in the channel, all the fluid has been removed through the channel walls. Hence, for suction,

$$0 \leq \eta \leq \frac{2\bar{u}(0)}{(\alpha + 1)V} = \frac{N_{Re}}{2(\alpha + 1)R}$$

Assuming the main channel flow to be laminar, equation (53) may be used to determine the ranges of R .

$$\frac{p(0, \lambda) - p(\eta, \lambda)}{\frac{1}{2} \rho \bar{u}(0)^2} = \frac{12(\alpha + 1)}{N_{Re}} \left[2 - \frac{27(\alpha + 1)R}{35} \right] \left[\frac{1}{\alpha + 1} - \frac{R}{N_{Re}} \cdot \eta \right] \cdot \eta \quad (53)$$

Since $(\eta) \leq \frac{N_{Re}}{2(a+1)R}$, the third factor is always positive. The pressure drop is always positive, so the second factor must be positive.

Hence,

$$2 \geq \frac{27(a+1)R}{35}$$

or

$$R \leq \frac{70}{27(a+1)}$$

Hence, the previous equations are valid with the following restriction:

(a) For suction,

$$0.0 < R \leq \frac{70}{27(a+1)} < N_{Re}$$

$$-1.0 < a \leq 1.0$$

$$0.0 \leq \frac{x}{h} \leq \frac{N_{Re}}{2(a+1)R}$$

(b) For injection,

$$R < 0$$

$$R < N_{Re}$$

$$-1.0 < a \leq 1.0$$

CHAPTER VI

DISCUSSION AND CONCLUSIONS

The Pressure Drop

Equation (53), the pressure drop, is plotted in Figure 2 through Figure 7 as a function of (η) for various values of N_{Re} . R and α are the parameters.

It is interesting to note the large effect of R on the pressure drop. At $N_{Re} = 500$ and $R = 0$ (a solid wall channel), the pressure drop is 4.8 at $(\eta) = 100$ (see Table 1). However, when $R = 1.0$ (suction) and $\alpha = 1.0$, the pressure drop is 0.658 which is an 86.4% decrease in the pressure drop. When $R = -1.0$ (injection) and $\alpha = 1.0$, the pressure drop is 11.9 which is a 148% increase in the pressure drop. Hence, a small value of R has a large effect on the pressure drop. It may be noted from Figure 2 through Figure 7 that at $R = \pm 1.0$ and $\alpha = 0.0$ the pressure drop is the same as that at $R = \pm 0.5$ and $\alpha = 1.0$ for a given value of N_{Re} .

Table 1 indicates the per cent increase or decrease in the pressure drop as a function of R , N_{Re} , and α .

When $\alpha = 1$ the solution for the pressure drop reduces exactly to the equation given by Berman⁴.

Figure 2 through Figure 7 show the following trends in the pressure drop:

- (a) An increase in N_{Re} causes a decrease in the pressure drop.
- (b) As R increases the pressure drop decreases.

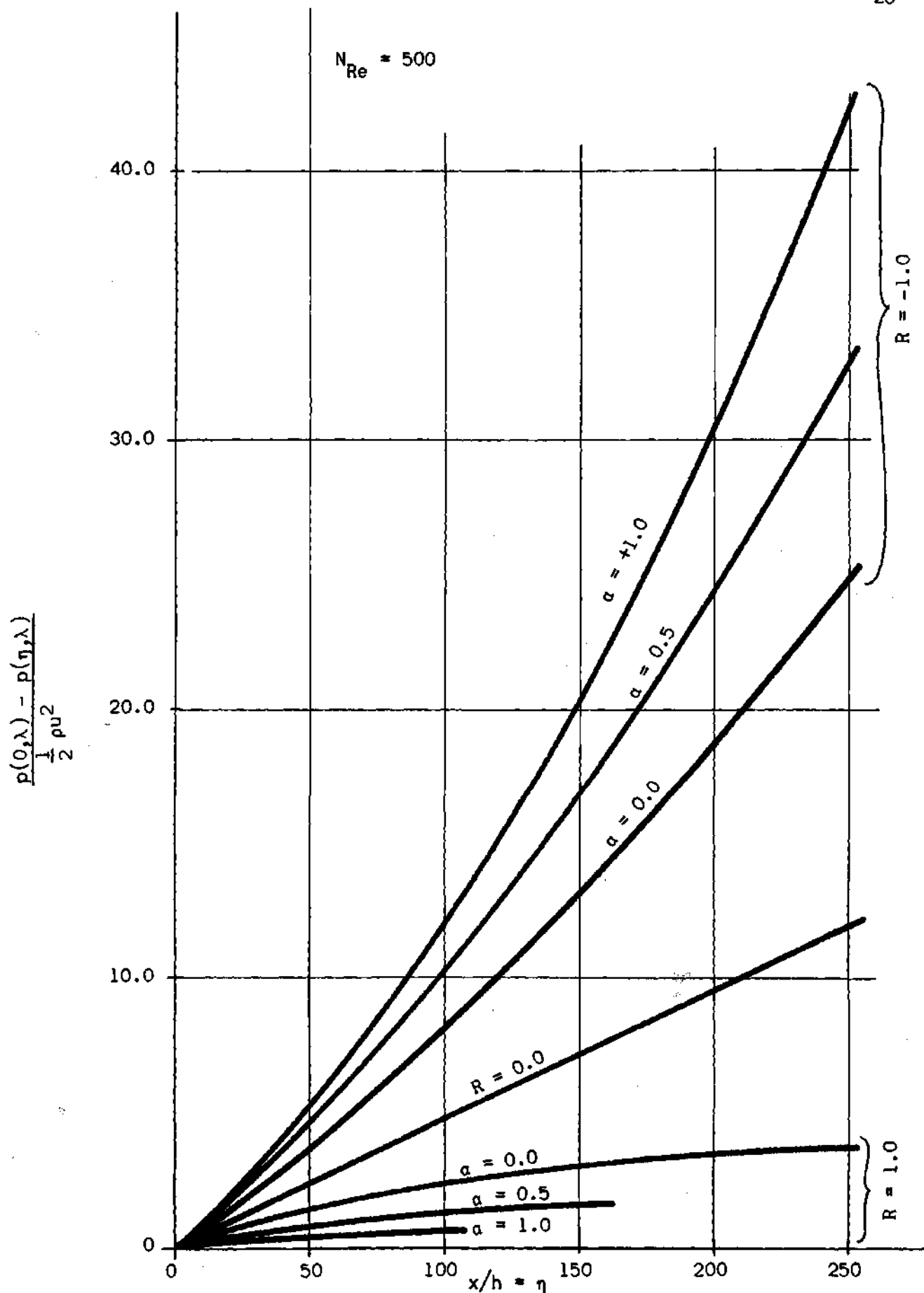


Figure 2. Pressure Drop Versus Distance Along Channel.

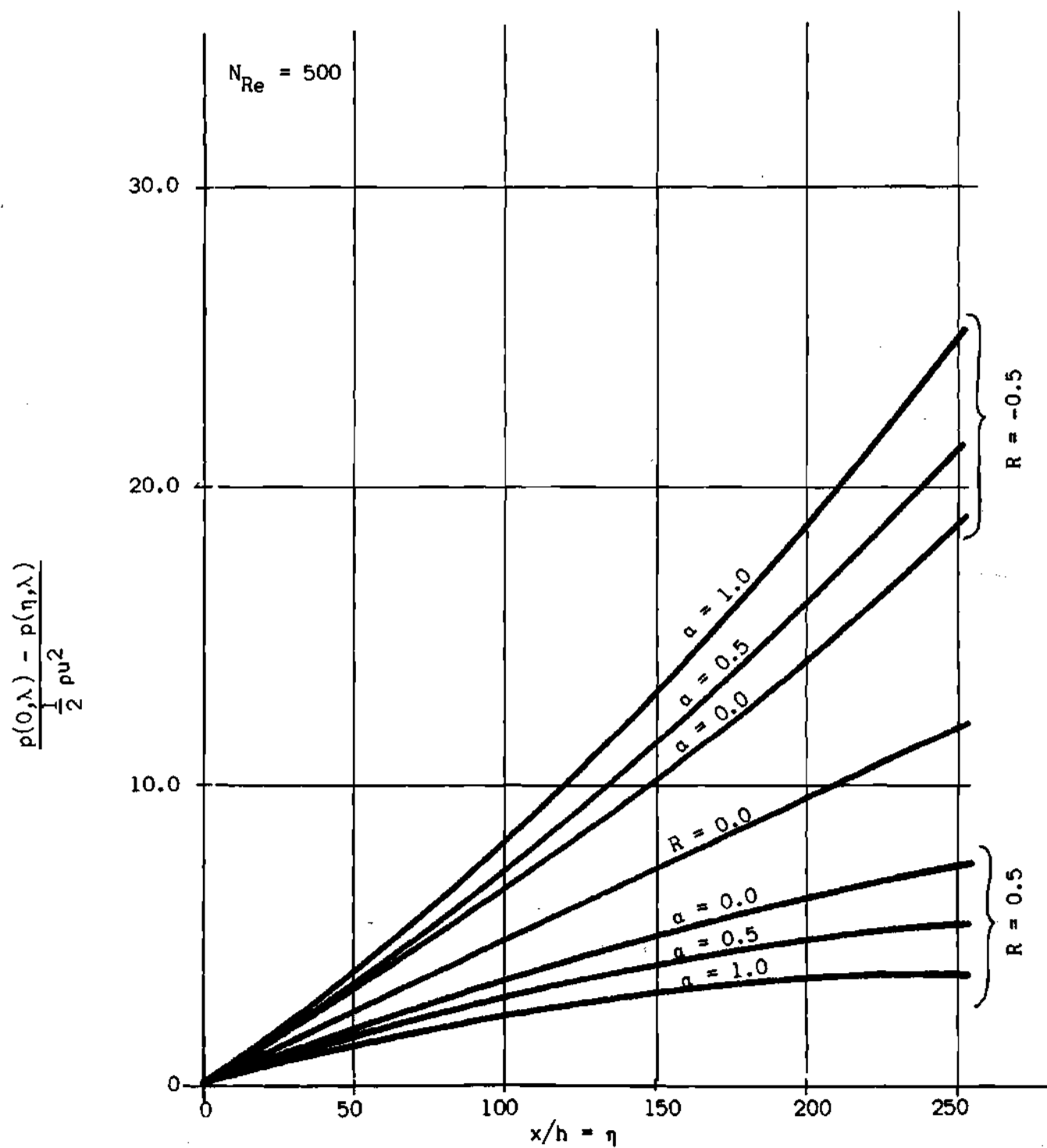


Figure 3. Pressure Drop Versus Distance Along Channel.

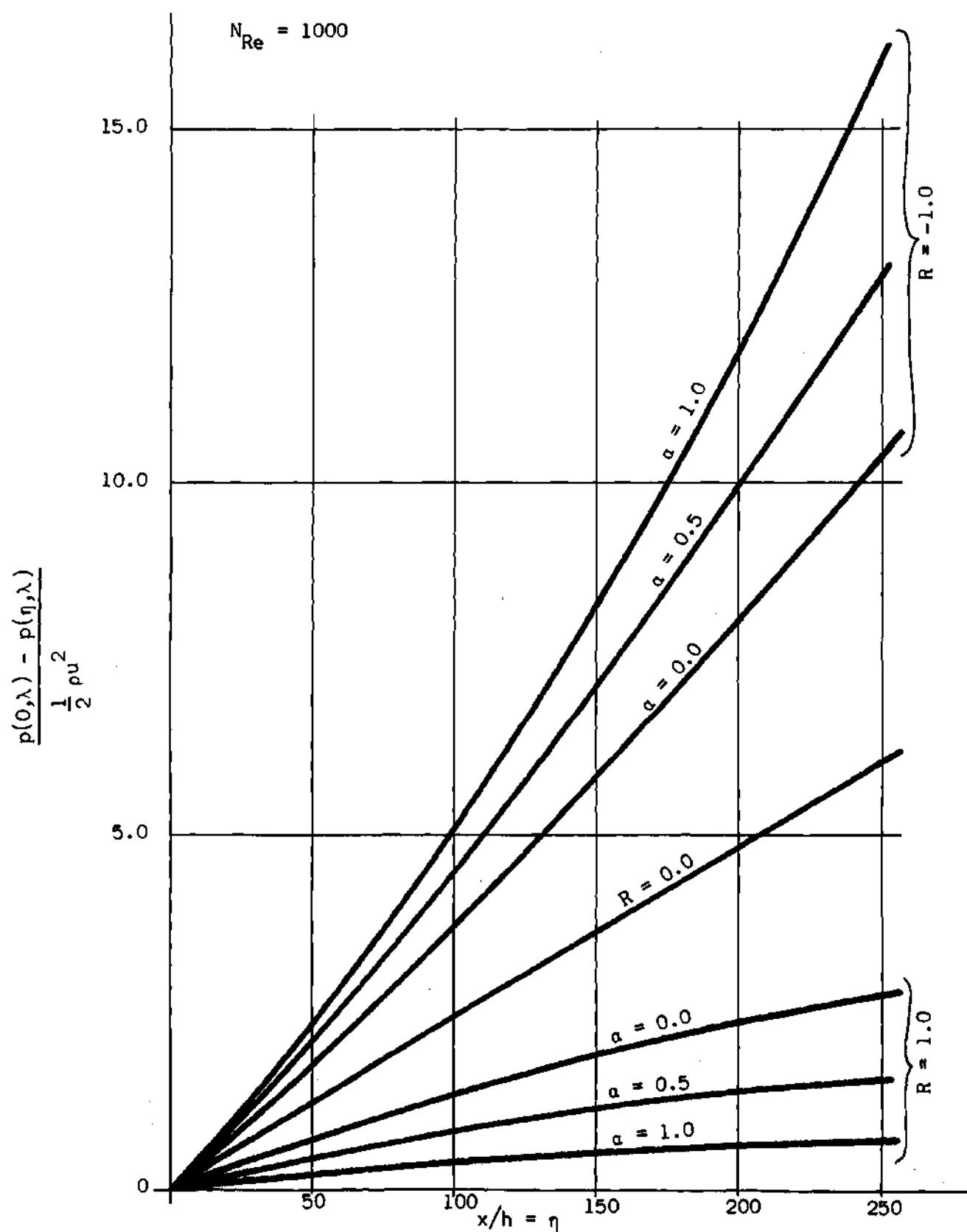


Figure 4. Pressure Drop Versus Distance Along Channel.

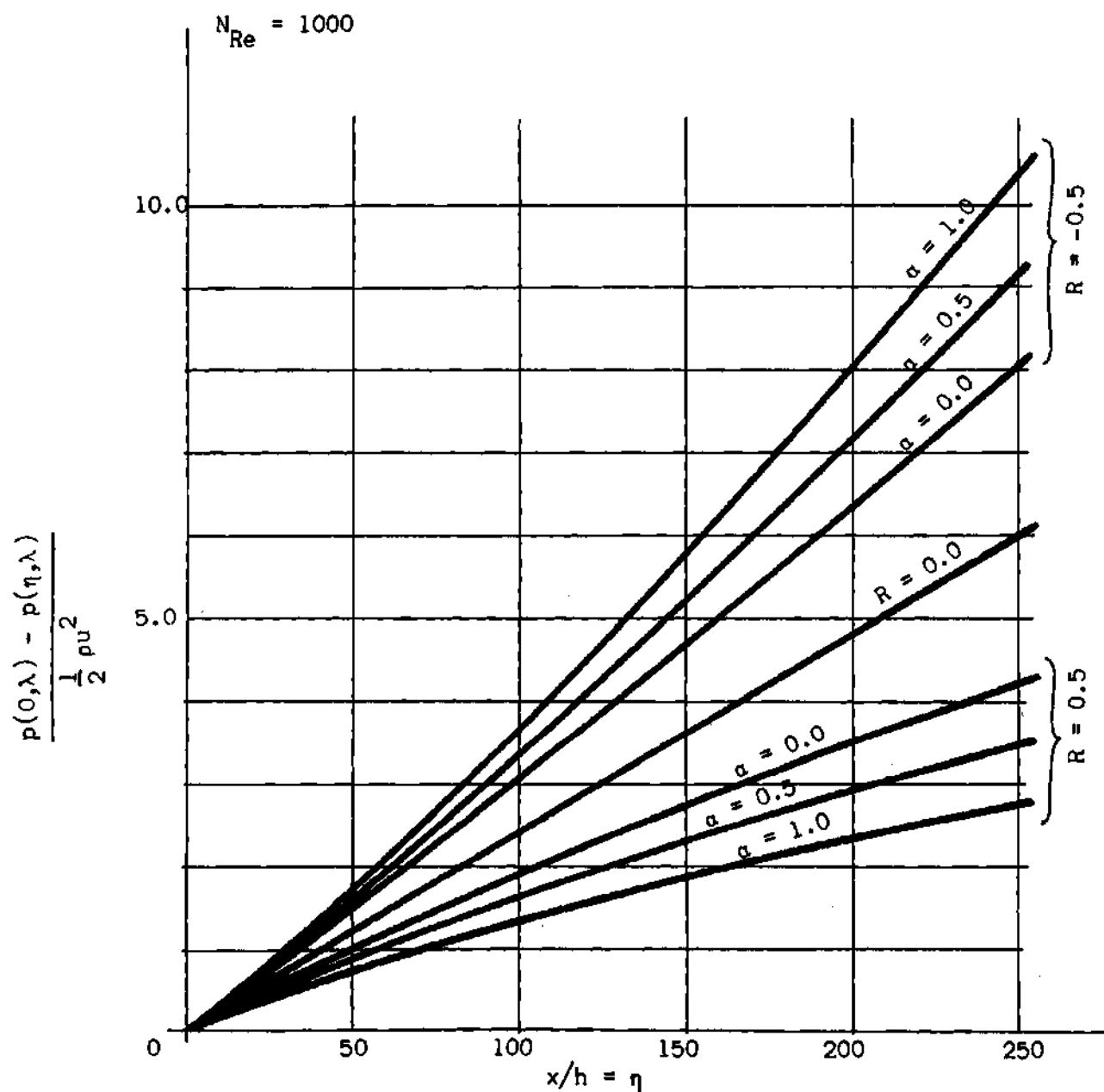


Figure 5. Pressure Drop Versus Distance Along Channel.

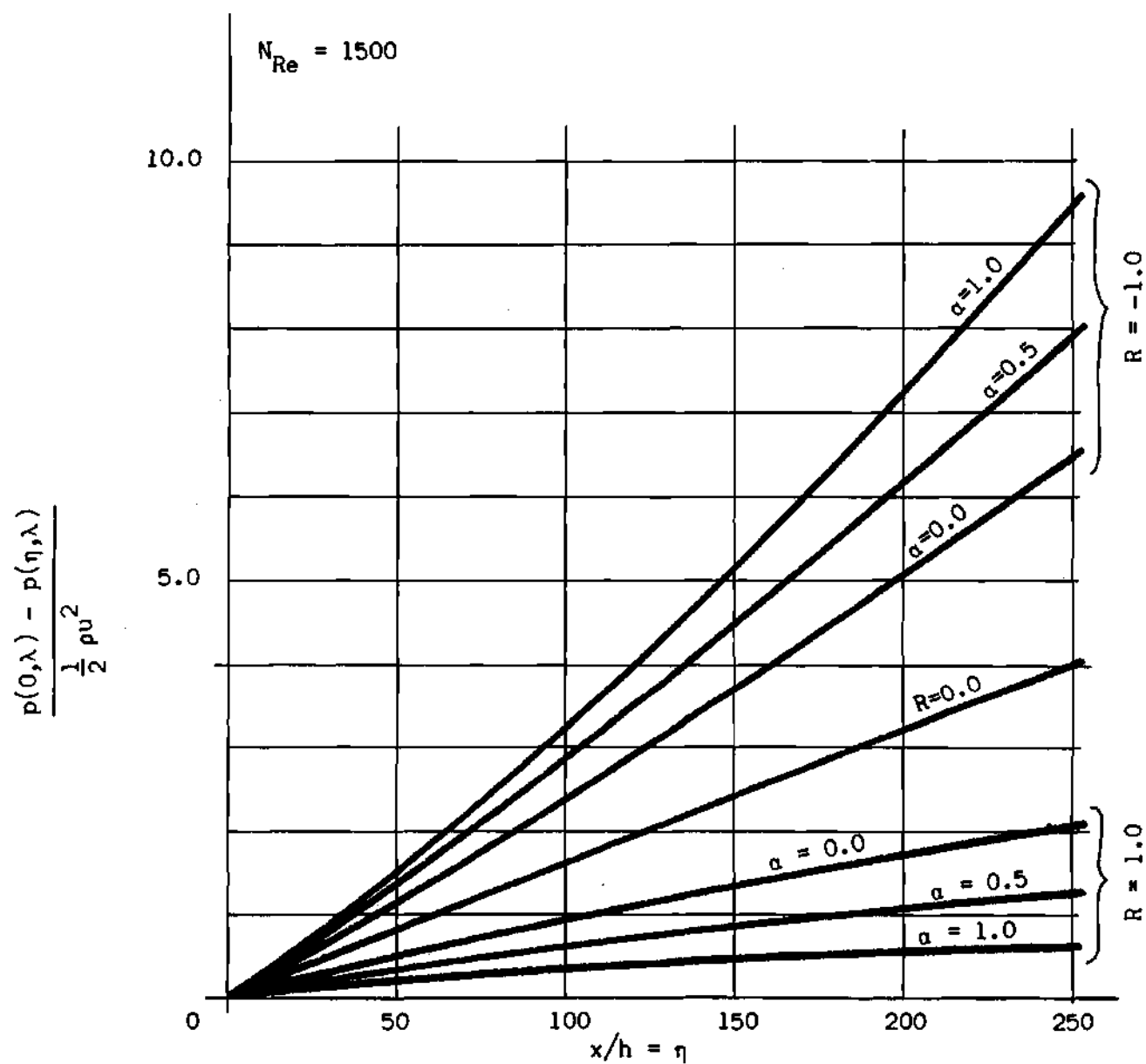


Figure 6. Pressure Drop Versus Distance Along Channel.

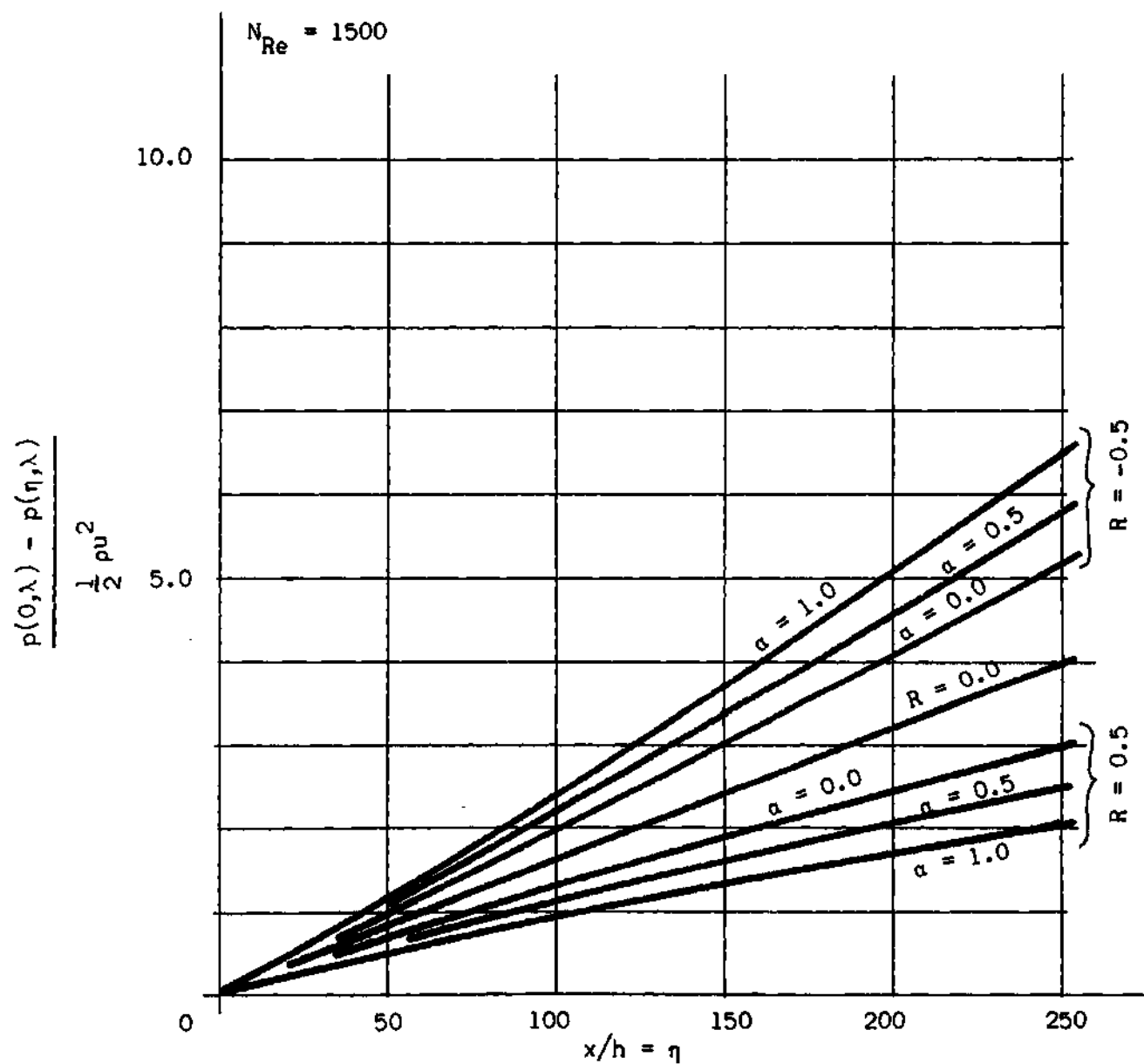


Figure 7. Pressure Drop Versus Distance Along Channel.

Table 1. Pressure Drop at $\eta = 100$

| N_{Re} | ΔP of a solid wall | R | α | ΔP of a porous wall | Injection % increase in ΔP | Suction % decrease in ΔP |
|----------|----------------------------|------|----------|-----------------------------|---------------------------------------|-------------------------------------|
| 500 | 4.8 | 1.0 | 1.0 | 0.658 | | 86.4 |
| | | | 0.5 | 1.416 | | 70.5 |
| | | | 0.0 | 2.359 | | 50.8 |
| | | 0.5 | 1.0 | 2.359 | | 50.8 |
| | | | 0.5 | 2.900 | | 39.6 |
| | | | 0.0 | 3.487 | | 28.4 |
| | | -0.5 | 1.0 | 7.982 | 66.2 | |
| | | | 0.5 | 7.117 | 48.3 | |
| | | | 0.0 | 6.298 | 31.2 | |
| | | -1.0 | 1.0 | 11.904 | 148.1 | |
| | | | 0.5 | 9.850 | 105.0 | |
| | | | 0.0 | 7.982 | 66.2 | |
| 1000 | 2.4 | 1.0 | 1.0 | 0.439 | | 81.7 |
| | | | 0.5 | 0.860 | | 64.1 |
| | | | 0.0 | 1.327 | | 44.7 |
| | | 0.5 | 1.0 | 1.327 | | 44.7 |
| | | | 0.5 | 1.578 | | 34.3 |
| | | | 0.0 | 1.840 | | 23.3 |
| | | -0.5 | 1.0 | 3.658 | 52.4 | |
| | | | 0.5 | 3.326 | 38.6 | |
| | | | 0.0 | 3.006 | 25.2 | |
| | | -1.0 | 1.0 | 5.102 | 112.5 | |
| | | | 0.5 | 4.357 | 81.5 | |
| | | | 0.0 | 3.658 | 52.4 | |

Table 1. (Continued)

| N_{Re} | ΔP of a solid wall | R | α | ΔP or a porous wall | Injection % increase in ΔP | Suction % decrease in ΔP |
|----------|----------------------------|------|----------|-----------------------------|---------------------------------------|-------------------------------------|
| 1500 | 1.6 | 1.0 | 1.0 | 0.317 | | 80.3 |
| | | | 0.5 | 0.607 | | 62.0 |
| | | | 0.0 | 0.917 | | 42.6 |
| | | 0.5 | 1.0 | 0.917 | | 42.6 |
| | | | 0.5 | 1.080 | | 32.5 |
| | | | 0.0 | 1.244 | | 22.2 |
| | | -0.5 | 1.0 | 2.365 | 47.8 | |
| | | | 0.5 | 2.166 | 35.4 | |
| | | | 0.0 | 1.972 | 23.2 | |
| | | -1.0 | 1.0 | 3.212 | 101.0 | |
| | | | 0.5 | 2.778 | 73.6 | |
| | | | 0.0 | 2.365 | 47.8 | |

- (c) At a constant value of N_{Re} and R , an increase in α will increase the pressure drop for injection and decrease the pressure drop for suction.

Velocity Profiles

The x and y -component velocity profiles are shown in Figure 8 through Figure 11 for various values of R .

Figures 8, 9, and 10 indicate that as α increases, the point at which the y -component of velocity becomes zero is moved above the center of the channel. Correspondingly, Figure 11 indicates that the point of maximum velocity in the x -direction is shifted above or below the centerline of the channel depending on the sign of R (the injection Reynolds number is negative) and the value of α . These effects are due to the injection or suction of fluid into or out of the main stream.

The Shear Stress

The large effects of R and α on the pressure drop are due to the increase (injection) or decrease (suction) of momentum and the change in viscous forces close to the wall of the channel.

Table 2 indicates the change in shear stress at the wall in a porous channel as compared to that in a solid wall channel for various values of α and R . The data was obtained from equation (64) for particular values of N_{Re} and (η) .

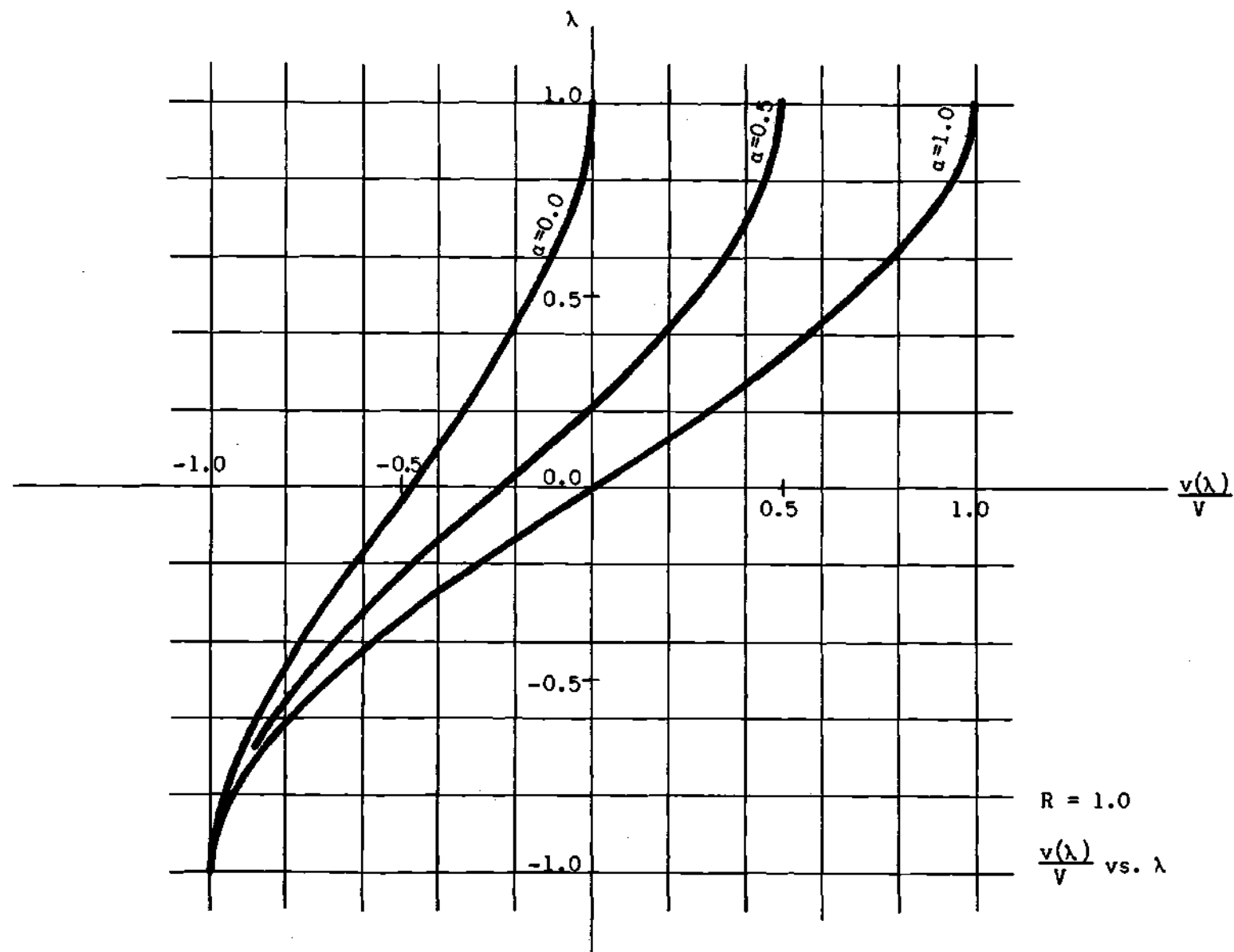


Figure 8. Velocity Ratio Versus Channel Width.

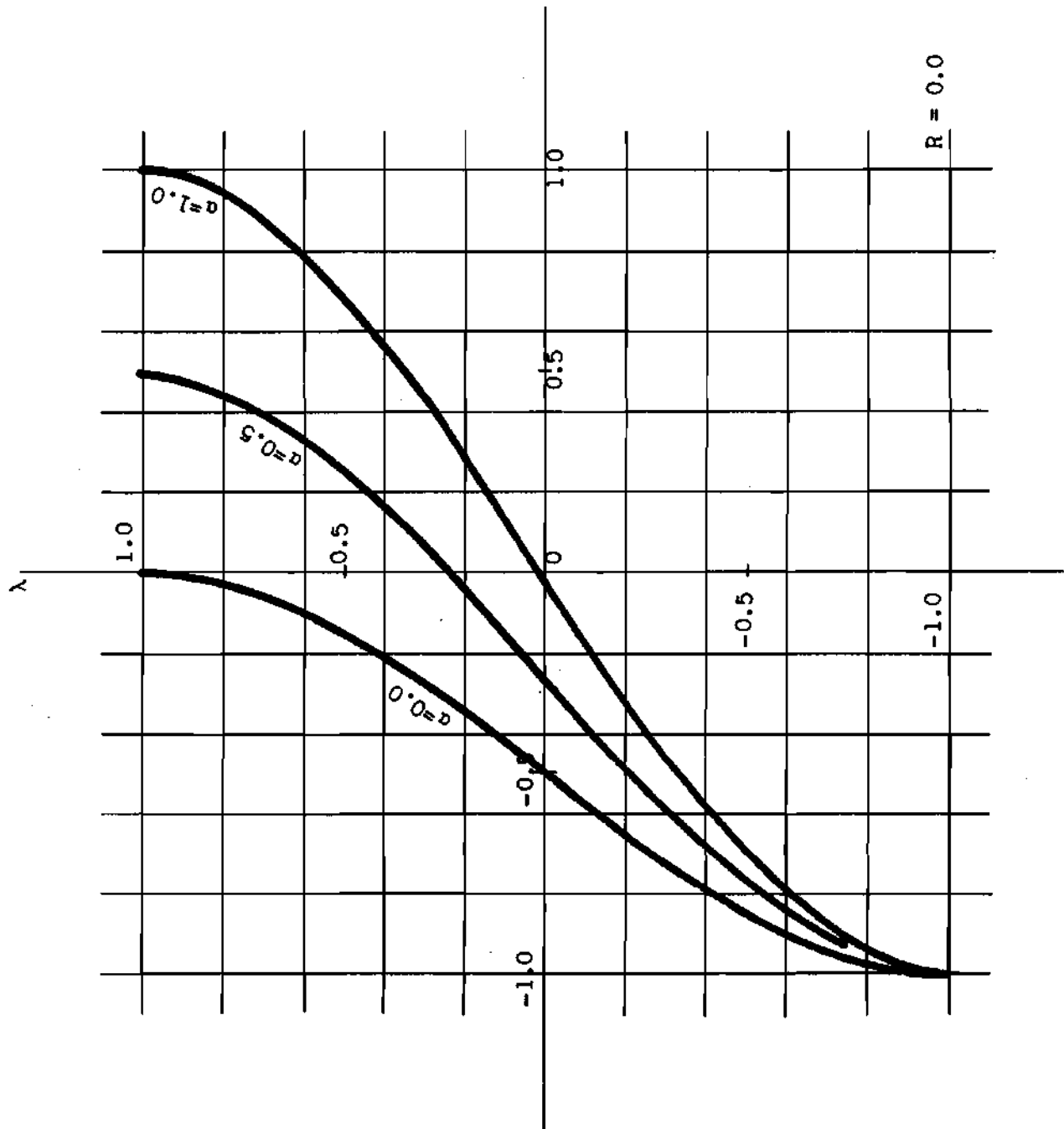


Figure 9. Velocity Ratio Versus Channel Width.

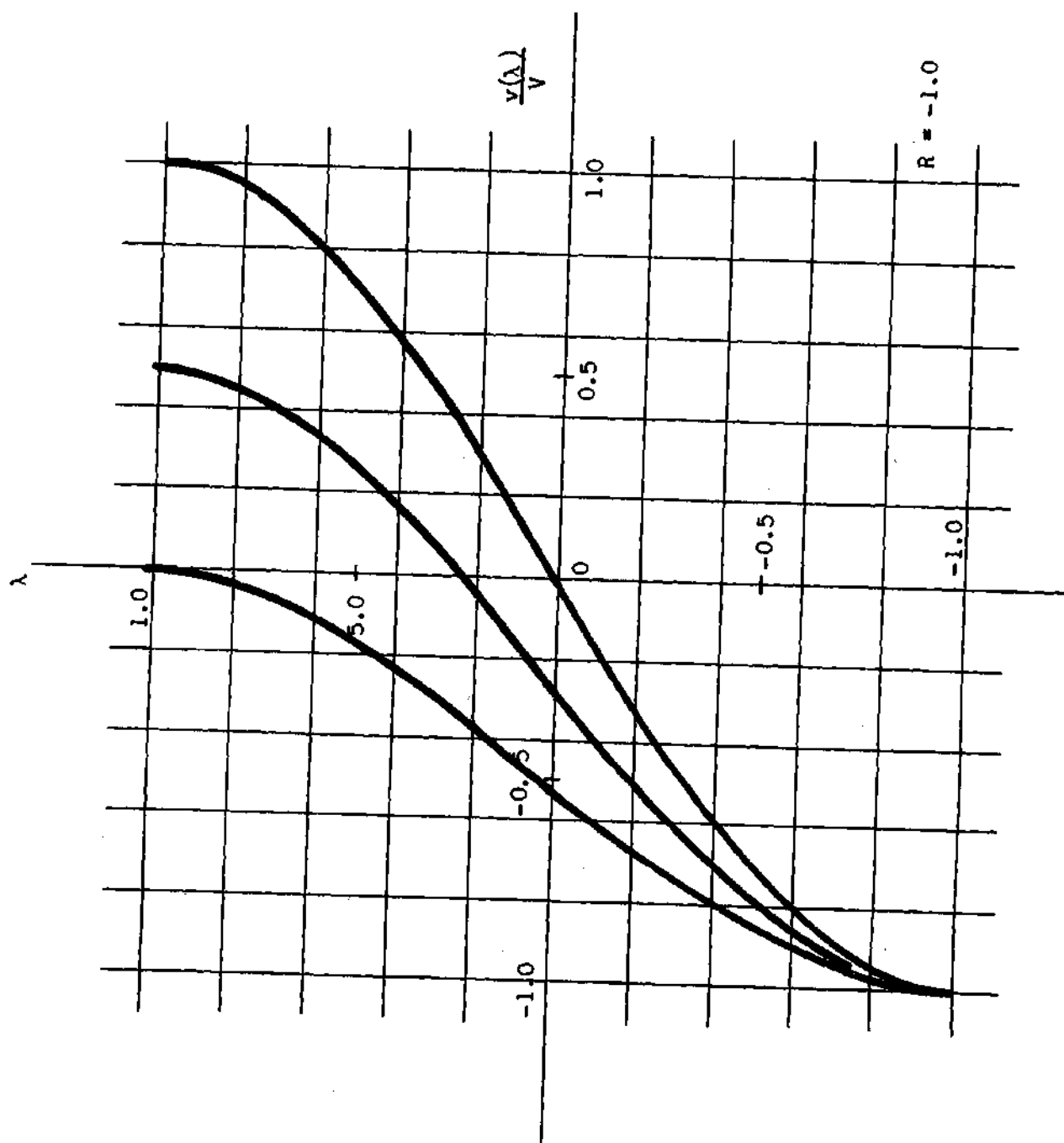


Figure 10. Velocity Ratio Versus Channel Width.

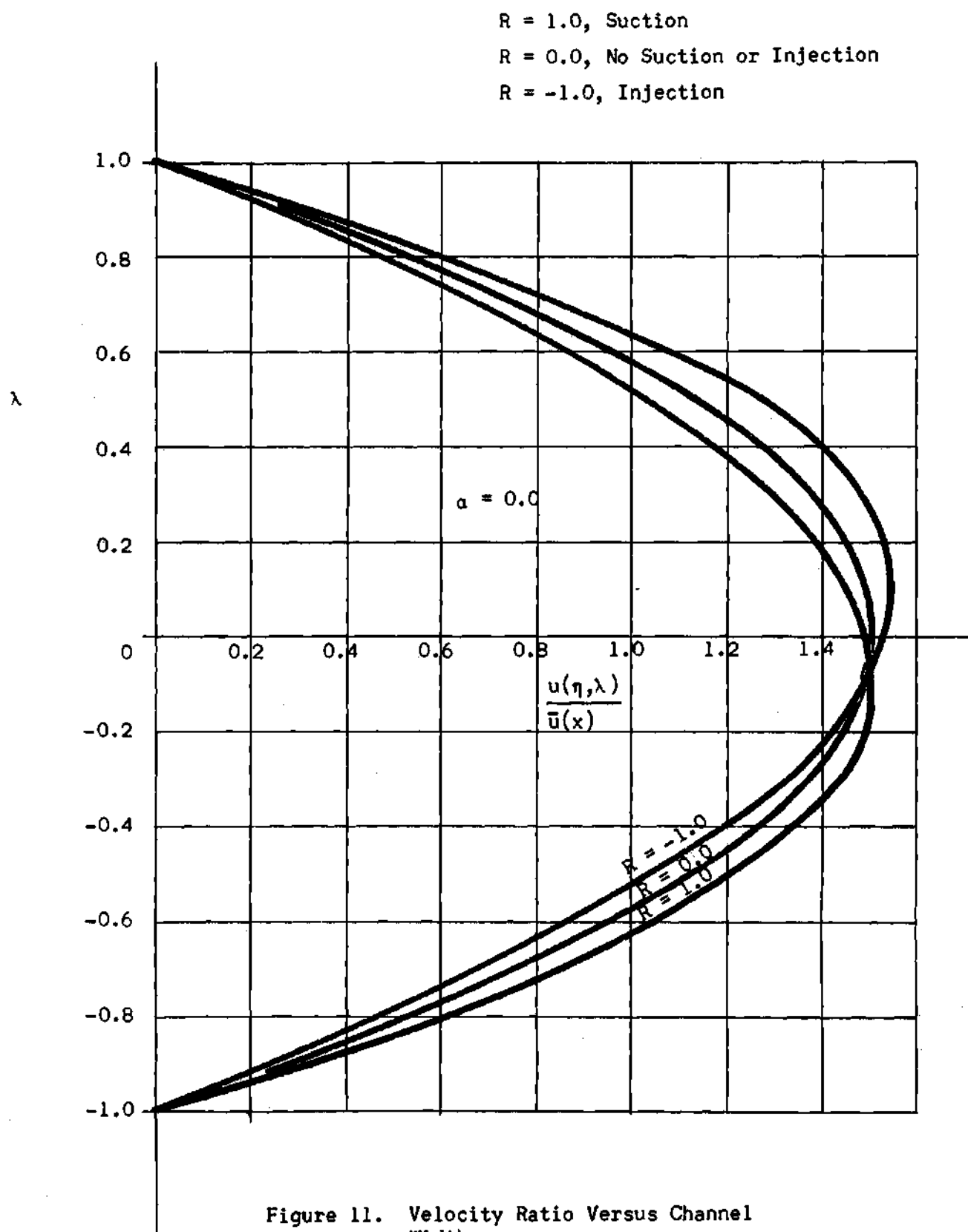


Table 2. Shear Stress

$$N_{Re} = 1000$$

$$\eta = 100$$

$$\lambda = 1.0$$

| a | R | $\frac{\tau}{\tau_0 \lambda=1.0}$ |
|----------------------|------|-----------------------------------|
| 1.0 | 1.0 | 0.800 |
| 0.0 | -1.0 | 0.983 |
| 0.0 | 1.0 | 0.945 |
| $-1.0 \leq a \leq 1$ | 0.0 | 1.000 |
| 1.0 | -1.0 | 1.200 |

APPENDIX

APPENDIX A

For detailed derivations of all equations and data presented in this research, please contact

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for a copy of Appendix A.

APPENDIX B

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